Lecture 18. (More inverse Laplace transforms ~~and applications to IVP's~~).

Last Time:

Derivatives in s-space: Suppose $f:[0,\infty)\to\mathbb{R}$ is piecewise continuous of exponential order α.

Then, for $s>\alpha$ and $n\in\mathbb{Z}_{\geq 0}$ $t^2\sin(t)$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \cdot \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}(s)\right).$$

The inverse Laplace Transform:

Suppose $F(s)$ is a function in a variable s.

If it exists, the inverse Laplace transform

$$\mathcal{L}^{-1}\{F(s)\}:[0,\infty)\longrightarrow\mathbb{R}$$

is the unique continuous function such that

if $f(t)=\mathcal{L}^{-1}\{F(s)\}(t)$ then $\mathcal{L}\{f(t)\}(s)=F(s)$.

(i)

Determine $\mathcal{L}^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$.

We write $\dfrac{3s+2}{s^2+2s+10}$

irreducible quadratic. "doesn't into linear terms with real coefficients"

$$= \frac{3s+2}{(s+1)^2+9}$$

$$= \frac{A(s+1)+B\cdot 3}{(s+1)^2+9}$$

(Partial fractions decomposition).

where A and B are constants.

Multiplying through by $\underline{(s+1)^2+9}$ gives

$$3s+2 = \underline{A(s+1)+B\cdot 3}.$$

Plug in $s=-1$ we obtain $B = -1/3$

Plug in $s=0$ we obtain $A = 2-3B$

$$= 3.$$

(ii)

So $$\frac{3s+2}{s^2+2s+10} = 3\cdot\frac{s+1}{(s+1)^2+3^2} - \frac{1}{3}\cdot\frac{3}{(s+1)^2+3^2}.$$

Hence $$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{3\cdot\frac{s+1}{(s+1)^2+3^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{(s+1)^2+3^2}\right\}(t)$$

$$= 3\cdot\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}(t) - \frac{1}{3}\underline{\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}(t)}$$

$$= 3\cdot e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t).$$

Example: Determine $\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t)$.

Consulting our table of Laplace transforms we recognize the denominator $(s+2)^4$ in the special case of the formula

$$\mathcal{L}\{e^{at}\cdot t^n\}(s) = \frac{n!}{(s-a)^{n+1}},$$

when $a = -2$ and $n = 3$.

Therefore $\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^{3+1}}\right\}(t) = e^{-2t}\cdot t^3$.

Applying the linearity properties of $\mathcal{L}$, we conclude

$$\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t) = \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\}(t)$$

$$= \frac{5}{6}\cdot e^{-2t}\cdot t^3.$$

Question: When does the inverse Laplace transform

$$\mathcal{L}^{-1}\{F(s)\}(t) \text{ exist ?}$$

Last Time: If $F(s)$ is not infinitely differentiable then there does not exists a continuous function $f:[0,\infty)\to\mathbb{R}$ such that $F(s) = \mathcal{L}\{f(t)\}(s)$.

So $F(s)$ must be infinitely differentiable in order to have a hope of $\mathcal{L}^{-1}\{F(s)\}(t)$ existing.

Theorem: Suppose $P(s)$ and $Q(s)$ are polynomial functions and assume

$$\deg(P(s)) < \deg(Q(s)).$$

If $F(s) = P(s)/Q(s)$ then $\mathcal{L}^{-1}\{F(s)\}(t)$ exists.

Proof Sketch: The proof is based on the existence of a partial fraction decomposition for the function $F(s) = P(s)/Q(s)$.

Recall: If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)$$

where $r_1, \dots, r_n$ are distinct real numbers, then $F(s)$ admits a partial fractions decomposition

$$F(s) = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \dots + \frac{A_n}{s-r_n}$$

where $A_1, \dots, A_n$ are real constants.

So $$\mathcal{L}^{-1}\{F(s)\}(t) = A_1 e^{r_1 t} + \dots + A_n e^{r_n t}.$$

Example: Determine $\mathcal{L}^{-1}\{F(s)\}(t)$ where

$$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}.$$

In this example $Q(s) = (s+1)(s+2)(s-3)$ factors into distinct linear factors over $\mathbb{R}$.

So $F(s)$ admits a partial fractions decomposition of the form

$$(*) \quad \frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3},$$

where A, B, and C are real constants.

To determine the constants A, B, and C multiply equation $(*)$ by $(s+1)(s+2)(s-3)$ to get

$$(\star) \quad 7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2).$$

By substituting $s = \underline{-1}$, $s = \underline{-2}$, and $s = \underline{3}$

into equation (✱) we obtain

$$7 \cdot (-1) - 1 = A \cdot (1) \cdot (-4) + B \cdot 0 + C \cdot 0,$$

$$7 \cdot (-2) - 1 = A \cdot 0 + B \cdot (-1) \cdot (-5) + C \cdot 0,$$

$$7 \cdot (3) - 1 = A \cdot 0 + B \cdot 0 + C \cdot 4 \cdot 5.$$

So $A = \underline{2}$, $B = \underline{-3}$, and $C = \underline{1}$.

Therefore $\mathcal{L}^{-1}\left\{ \dfrac{7s - 1}{(s+1)(s+2)(s-3)} \right\}(t)$

$$= \mathcal{L}^{-1}\left\{ \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3} \right\}(t)$$

$$= 2e^{-t} - 3e^{-2t} + e^{3t}.$$

Question: Suppose $P(s)$ and $Q(s)$ are polynomial functions with $\deg(P) < \deg(Q)$.

How do we calculate $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$ in the case when (i) Q has a repeated linear factor.

(ii) Q has an irreducible quadratic factor.

(i) The case of a repeated linear factor.

If $s-r$ is a factor of $Q(s)$ and $(s-r)^m$ is the highest power of $s-r$ that divides $Q(s)$ then the partial fraction decomposition of $P(s)/Q(s)$ contains a term of the form

$$\frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \dots + \frac{A_m}{(s-r)^m}$$

where $A_1, \dots, A_m$ are constants.

Example: Determine $\mathcal{L}^{-1}\left\{ \dfrac{s^2+9s+2}{(s-1)^2\cdot(s+3)} \right\}(t)$?

In this example $Q(s) = \underline{(s-1)^2\cdot(s+3)}$ contains $(s-1)$ as a repeated linear factor.

So $\dfrac{s^2+9s+2}{Q(s)}$ admits a partial fractions decomposition of the form

$$(*) \qquad \frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

where A, B, and C are constants.

To determine the constants A, B, and C multiply equation $(*)$ by $\underline{(s-1)^2\cdot(s+7)}$ to get

$$(\star) \qquad s^2+9s+2 = A(s-1)(s+3) + B\cdot(s+3) + C\cdot(s-1)^2.$$

By substituting $s=1$ and $s=-3$ into equation $(\star)$ we obtain

$$1^2 + 9\cdot 1 + 2 = A\cdot 0 + B\cdot 4 + C\cdot 0^2$$

and $$(-3)^2 + 9\cdot(-3) + 2 = A\cdot 0 + B\cdot 0 + C\cdot(-4)^2.$$

So $B = \underline{3}$ and $C = \underline{-1}$.

To determine A we substitute $s=0$ into equation $(\star)$ which yields

$$0^2 + 9\cdot 0 + 2 = A\cdot(-3) + B\cdot 3 + C\cdot 1$$

Since $B=3$ and $C=-1$ we conclude that
$3A = 3B + C - 2 \Rightarrow A = \frac{1}{3}(9-1-2)$

$A = \underline{2}$. Hence

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}\right\}(t)$$

$$= 2\cdot \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + 3\cdot \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)$$

$$= 2\cdot e^{t} \qquad + 3\cdot t\cdot e^{t} \qquad - e^{-3t}.$$

(ii) <u>The case of irreducible quadratic factors.</u>

If $(s-\alpha)^2+\beta^2$ is a factor of $Q(s)$ that cannot be <u>reduced</u> to linear factors with coefficients in $\mathbb{R}$ and m is the $\cancel{(s-\alpha)^2+\beta^2}$ highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$ then the partial fractions decomposition of $Q(s)$ contains a term of the form

$$\frac{A_1(s-\alpha)+\beta\cdot B_1}{(s-\alpha)^2+\beta^2} + \cdots + \frac{A_m(s-\alpha)+\beta\cdot B_m}{\left[(s-\alpha)^2+\beta^2\right]^m}$$

where $A_1, B_1, \ldots, A_m, B_m$ are <u>constants</u>.

<u>Example:</u> Determine $\mathcal{L}^{-1}\left\{ \frac{2s^2+10s}{(s^2-2s+5)(s+1)} \right\}(t)$.

In this example $Q(s) = \underline{(s^2-2s+5)(s+1)}$ contains s^2-2s+5 as an <u>irreducible</u> quadratic factor.

Since $s^2-2s+5 = \underbrace{s^2-2s+1}+4$

$= (s-1)^2 + 2^2$, $\alpha = 1$, $\beta = 2$. $m = 1$.

we express the partial fractions decomposition as

$$(*) \quad \frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \underline{\frac{A\cdot(s-1)+B\cdot 2}{(s-1)^2+2^2} + \frac{C}{s+1}}.$$

where A, B, and C are constants.

Multiplying equation (*) by $\underline{(s^2-2s+5)(s+1)}$

gives

$$(\star) \quad 2s^2+10s = [A\cdot(s-1)+B\cdot 2](s+1) + C\cdot(s^2-2s+5)$$

To determine the constants A, B, and C we begin by substitute $s = -1$ into $(*)$.

This gives $2 - 10 = [A \cdot (-2) + 2 \cdot B](0) + 8C$.

Hence $C = \underline{-1}$.

Next we substitute in $s = 1$ to obtain

$$2 \cdot 1^2 + 10 \cdot 1 = [A \cdot 0 + 2B](2) + 4C.$$

$4B + 4C = 12$.

Hence $B = \underline{4}$.

Finally we substitute in $\underline{s = 0}$ to obtain

$$0 = [A \cdot (-1) + 2B](1) + 5C$$

which gives

$$A = 2B + 5C$$
$$= 3$$

So $$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} = \frac{3(s-1) + 2 \cdot (4)}{(s-1)^2 + 2^2} - \frac{1}{s+1}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}(t)$$

$$=\mathcal{L}^{-1}\left\{\frac{3(s-1)+4\cdot 2}{(s-1)^2+2^2}-\frac{1}{s+1}\right\}(t)$$

$$=3\cdot\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}(t)$$

$$+4\cdot\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}(t)$$

$$-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)$$

$$=3e^t\cos(2t)+4e^t\sin(2t)-e^{-t}.$$